Lecture 3 — Existence of Analytic solutions — The Method of Majorants

Definition 1. *f, $F \in C^\infty(I)$ where $I = (c-\epsilon, c+\epsilon)$. We say f is majorized at c by F if*

$$|f^{(k)}(c)| \le F^{(k)}(c)$$

and we write $f \ll F$.

Lemma 1. *Suppose $f \in G^1_{M,r}(0)$ then $f \ll \phi$ at 0, where*

$$\phi(x) = \frac{Mr}{r-x}.$$

If $f(0) = 0$ then $\phi(x) = \frac{Mr}{r-x} - M$.

proof. We have $f \in G^1_{M,r}(0) \Leftrightarrow |f^{(k)}(0)| \le \frac{Mk!}{r^k}$. Computing derivatives of ϕ we have $\phi'(x) = \frac{Mr}{(r-x)^2}, ..., \phi^{(k)}(x) = \frac{Mrk!}{(r-x)^{k+1}}$. Evaluated at $x = 0 \implies \phi^{(k)}(0) = \frac{Mrk!}{r^{k+1}}$.

$$\therefore\ |f^{(k)}(0)| \le \frac{Mk!}{r^k} = \frac{Mrk!}{r^{k+1}}.$$

□

(ODE) Case: Consider Cauchy problem

$$v' = g(v), \qquad v(0) = \alpha \qquad g \in G^1_{M,r}(\Omega).$$

In our attempt to prove the existence of analytic solution, we will construct a problem which behaves worse but can be solved explicitly. Consider

$$u(t) = v(t) - \alpha$$

$$u' = g(u+\alpha) =: f(u)$$

FIG 2.1.
Suppose $u(t) = v(t) - \alpha - tv'(t)$

$$\implies u'(0) = 0$$

$$\implies f(0) = 0.$$

$u'(t) = f(u(t)), \qquad u(0) = 0$
$u'' = f'(u)u' \qquad u''' = f''(u)(u')^2 + f'(u)u''$ ...

$$[f(u)]^{(k)} = q_k(f'(u), ..., f^{k-1}(u), u, ..., u^k)$$

with positive polynomial coefficients.

$$u^{(k+1)}(0) = q_k(f'(0), ..., f^k(0), u'(0), ..., u^k(0))$$

Class notes by Ibrahim Al Balushi

We notice that u^k and u^{k+1} are expressed recursively hence we reduce order that way to gain new poly Q

$$= Q_k(f(0), f'(0), ..., f^k(0))$$

with positive coefficients.
Assume $f \ll F$ at 0

$$U' = F(U), \qquad U(0) = 0$$

$|u^{(k+1)}(0)| = |Q_k(f(0), ..., f^{(k)}(0))| \leq Q_k(|f(0)|, ..., |f^{(k)}|) \leq Q_k(F(0), ..., F^{(k)}(0)) = U^{(k+1)}(0)$
$\implies u \ll U$ at 0.
By assumption $f \in G^1_{M,r}(0) \implies f \ll \phi, \phi = \frac{Mr}{r-x}$. We have $U' = \frac{Mr}{r-U}$,

$$dU(r - U) = Mrdt \implies rU = \frac{1}{2}U^2 + Mrt + C \ (C = 0)$$

$\implies U(t) = r - \sqrt{r^2 - 2Mrt}$ where we chose the negative root to ensure $u(0) = 0$ remains true. In general $r\sqrt{1-a} = r(1 - \frac{1}{2}a - a^2 - ...)$ taylor expansion, so $U(t) = c_1t + c_2t^2 + ...(c_j \geq 0)$ is analytic therefore $u(t)$ analytic.

Case (System of ODE's) $\quad u_j'(t) = f_j(u_1(t), ..., u_n(t)) \qquad j = 1...n.$

Definition 2. *$f : \mathbb{R}^n \to \mathbb{R}$ is (real) analytic at $c \in \mathbb{R}^n$ if*

$$(*) \qquad f(x) = \sum_{\alpha_1,...,\alpha_n \geq 0} a_{\alpha_1...,\alpha_n}(x_1 - c_1)^{\alpha_1}...(x_n - c_n)^{\alpha_n}$$

shorthand $= \sum_{|\alpha|\geq 0} a_\alpha (x - c)^\alpha$ *in a nbhd of c.*

Suppose $(*)$ converges for some x then,

$$|a_\alpha||x_1 - c_1|^{\alpha_1}...|x_n - c_n|^{\alpha_n} \to 0$$

as $|\alpha| \to \infty$ *i.e* $\exists M \in \mathbb{R}$ *s.t*

$$|a_\alpha||x_1 - c_1|^{\alpha_1}...|x_n - c_n|^{\alpha_n} \leq M < \infty$$

FIG 2.2.

$$Q_r(c) = \{x \in \mathbb{R}^n \ : \ |x_i - c_i| < r, \forall i\}.$$

$\mathbb{R}^n$ cube.
Assume $\quad a_\alpha r^{|\alpha|} \leq M < \infty \quad \forall\alpha$. Then (α) converges in $Q_r(c)$

$$|a_\alpha||x_1 - c_1|^{\alpha_1}...|x_n - c_n|^{\alpha_n} \leq |a_n|\rho^{|\alpha|}\frac{|x_1 - c_1|^{\alpha_1}}{\rho^{\alpha_1}}...\frac{|x_n - c_n|^{\alpha_n}}{\rho^{\alpha_n}} \leq |a_\alpha|\rho^{|\alpha|}\delta^{|\alpha|} \leq M\delta^{|\alpha|} \qquad (\delta < 1)$$

i.e if power series $(*)$ converges at pt c, then $\exists$nbhd of c *s.t.* $\forall x \in$nbhd(c), $(*)$ converges.

Definition 3. *Let $f, F \in C^\infty(\mathbb{R}^n)$. $f \ll F$ if*

$$|\partial^\alpha f(c)| \leq \partial^\alpha F(c)$$

where $\partial^\alpha f = \partial_1^{\alpha_1}...\partial_n^{\alpha_n} f$

$$u_j' = f_j(u), \qquad u_j(0) = 0.$$
$$u_j'' = \partial_k f_j(u) u_k', \qquad u_j''' = \partial_l \partial_k f(u) u_k' u_l' + \partial_k f_j(u) u_k''.$$

$$u_j^{(k+1)} = [f_j(u)]^{(k)} = q_k(u', ..., u^{(k)}, \{\partial^\alpha f_j\}) \qquad (|\alpha| \leq k) = Q_k(\{\partial^\alpha f_j(0)\}).$$

Suppose

$$f_j \ll F_j. \qquad U_j' = F(U), \qquad U_j(0) = 0.$$

$|u_j^{k+1}(0)| = |Q_k(\{\partial^\alpha f_j(0)\})| \leq Q_k(\{|\partial^\alpha f_j(0)|\}) \leq Q_k(\{\partial^\alpha F_j(0)\}) = U_j^{k+1}(0). \implies u_j \ll U_j.$
$f \in G^1_{M,r}(0)\ :\ |\partial^\alpha f(0) \leq M|\alpha|! r^{-|\alpha|}$, then $f \ll \phi\ ;\ \phi = \frac{Mr}{r-(x_1+...+x_n)}$.

$$f_j \in G^1_{M,r}(0). \qquad U_j = \frac{Mr}{r-(U_1+...+U_n)}, \qquad U_j(0) = 0.$$

$$U_1 = U_2 = ...U_n = U$$

$U' = \frac{Mr}{r-nU} \implies U(t) = \frac{r-\sqrt{r^2-2nMrt}}{n}$.

What we have shown was the existence of an analytic solution for a system of ODE's.

Theorem 1 (Cauchy-Kovalesky)**.** $\partial_n u_j = f_j(x, u, \partial_1 u, ...\partial_{n-1} u) \quad u_j(x_1, ..x_{n-1,0}) = \phi_j(x_1, ...x_{n-1})$ *are all analytic. Then there exists a unique analytic solution in nbhd of 0.*